

# Deformed fields and Moyal construction of deformed super Virasoro algebra

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## Abstract

Studied is the deformation of super Virasoro algebra proposed by Belov and Chaltikhian. Starting from abstract realizations in terms of the FFZ type generators, various connections of them to other realizations are shown, especially to deformed field representations, whose bosonic part generator is recently reported as a deformed string theory on a noncommutative world-sheet. The deformed Virasoro generators can also be expressed in terms of ordinary free fields in a highly nontrivial way.

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# 1 Introduction

The deformed Virasoro algebras proposed in an early stage [1]- [5] have recently become more suitable to be examined in physical connections. These algebras were studied originally from the purely mathematical motivation to seek a  $q$ -analogue of the Virasoro algebra in the context of quantum groups. However, as far as these algebras are concerned, it has become more appropriate to discuss in some physical connections than in the quantum group context; for example, in string theory [6], a solvable lattice model [7], the lowest Landau levels [8, 9], and soliton systems [10].

Among them a bosonic string on a noncommutative geometry has recently been constructed [6], and its string action possesses the deformed Virasoro algebra as a symmetry, like the Virasoro algebra for the ordinary bosonic string (on commutative world-sheets). The field equation obeys a discrete time evolution, and the oscillators satisfy the usual  $q$ -deformed Heisenberg algebra,

$$[a_n, a_m] = \frac{q^n - q^{-n}}{q - q^{-1}} \delta_{n+m,0} . \quad (1.1)$$

Noncommutative geometry is suggested to appear in a particular lowest energy limit of string theory in a constant background  $B$ -field configuration [11, 12]. This situation is similar to the appearance of noncommutative discrete translations (magnetic translations [13]) for electron systems confined on a two-dimensional surface in a constant magnetic field. In these lowest Landau level systems, the same deformed Virasoro algebra is realized [8] as a particular combination of the Fairlie-Fletcher-Zachos (FFZ) algebra [14] (or algebra of magnetic translations). The discrete nature of a magnetic lattice is also a resemblance to the discrete time prescription of the above noncommutative string theory. In fact, the FFZ type realization implies an intimate relation to the Moyal type deformation [15], which has recently been paid much attention in the context of noncommutative string/field theories [16].

In order to find the corresponding deformed symmetry of a superstring on a noncommutative geometry in view of the above connection, it is worth to first realize a supersymmetric extension of the deformed Virasoro algebra in terms of the FFZ generators. In this paper, we focus our attention on the superalgebra proposed by Belov and Chaltikhian (BC) [4]. This superalgebra is in fact a supersymmetric extension of the symmetry in the deformed bosonic string [6]. We shall follow the same methods as developed in a different type of deformed super Virasoro algebra [8, 17, 18], where bosonic and fermionic parts participate in an asymmetric way. In the present case, they are symmetric, and this is certainly an advantageous point. However, magnetic translations are nothing but differential operators ( $q$ -difference operators), and the deformed super Virasoro algebra

obtained in this way becomes a centerless algebra. From the viewpoint of applications to field theories, we hence have to make a connection from the centerless realizations to a field realization. Hence, starting from the most general formulae, we shall derive various realizations and their related formulae extensively in terms of differential operators, matrices, ordinary/deformed fields (boson, fermion, and ghosts).

One of nontrivial issues of the paper is the relation between ordinary and deformed fields. In this paper we introduce the deformed boson field which is comprised of the  $q$ -boson oscillators satisfying the relation (1.1). (We also introduce a similar deformed fermion.) Taking account of normalization changes in (1.1), the ( $q$ -boson) oscillators look equivalent to the usual boson oscillators. However, as suggested in [6], the parameter  $q$  possesses the clear meaning of a time discretization on a world-sheet, emerged from a noncommutative geometry. We shall also support the nontriviality of the relation (1.1) from a different viewpoint. Although the commutation relation itself may be trivial in the sense of normalization, it is no longer true at the level of a field object. We suggest the ordinary field realizations which cannot be connected to the deformed fields by the normalization changes. Even if connected to the deformed fields in such a way, the deformed fields become nonlocal objects which require fractional differential calculus. Although the present formulation is not directly related to noncommutative geometry, the symmetry generator form is exactly the same as in the deformed bosonic string [6] (see also [7], [18, 19]). There should be clear connections among them, and hence it is very useful to study the corresponding superalgebra.

The deformed Virasoro algebra may also play an important role in integrable systems. To reveal the nature of integrability, it is of course necessary to understand what type of deformation maintains the integrability in each system. Otherwise pathological deformation tends to destroy integrability leading to a chaotic behavior. From this view point, magnetic translations are suitable mathematical means of describing an integrable discretization, since it is a well-behaving difference operator on a lattice. In fact, the deformed Virasoro algebra (noncommutative magnetic translation) on a magnetic lattice becomes the Virasoro algebra (commutative continuous translation) as a magnetic field vanishes. (This is exactly the  $q \rightarrow 1$  limit [20] in the terminology of quantum groups.) We hence expect that a system possessing the deformed Virasoro symmetry will be related to its integrability in somehow algebraic way. Similarly, the investigation on this symmetry might provide novel suggestion in various related areas as well as in string theory.

This paper is organized as follows. In Section 2, we put necessary formulae and brief comments

on the BC superalgebra. Section 3 is an entirely new part, where we discuss the deformed field realizations of the BC superalgebra. We present bilinear integral forms, adjoint commutator representations and their matrix forms. In Section 4, we show two FFZ realizations of the (centerless) BC superalgebra. When describing them by differential operators and the Pauli matrices, both realizations are organized into the similar (but slightly different) matrix forms presented in Section 3. In Section 5, we improve this different matrix structure by introducing another set of the FFZ generators as well as a particular noncommutative generalization of the Pauli matrices. We show four classes of the realizations of this type. Each class is an infinite set represented by one parameter  $\Delta$ . Among them we only discuss two specific cases, which exactly reduce to the same matrix forms as presented in Section 3. These are discussed case by case. It is also shown that the familiar differential realizations in superspace are obtained in the limit of  $q \rightarrow 1$ . In Section 6, we discuss the realizations by ordinary free fields. Their relations to the deformed field realizations are highly nontrivial. Section 7 concerns (super) ghost field realizations. We obtain the same copies of the BC algebra with new central extensions. Section 8 contains conclusions and discussions.

## 2 The Belov-Chaltikhian (BC) superalgebra

The superalgebra proposed by Belov and Chaltikhian [4] is an algebra with two sets of indices<sup>1</sup>, and organized in the following way (see [18] for more details):

$$[L_n^{(k)}, L_m^{(l)}] = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \left[ \frac{n(\varepsilon l + 1) - m(\eta k + 1)}{2} \right]_- L_{n+m}^{(\varepsilon k + \eta l + \varepsilon \eta)} + C \delta_{n+m, 0} . \quad (2.1)$$

$$[L_n^{(k)}, G_m^{(l)}] = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \varepsilon \left( \frac{2}{q - q^{-1}} \right)^{\frac{1-\varepsilon\eta}{2}} \left[ \frac{n(l + \eta) - m(\varepsilon k + \eta)}{2} \right]_{-\varepsilon\eta} G_{n+m}^{(l + \varepsilon k + \eta)} , \quad (2.2)$$

$$\{G_r^{(k)}, G_s^{(l)}\} = \sum_{\varepsilon, \eta = \pm 1} \left( \frac{q - q^{-1}}{2} \right)^{\frac{1-\varepsilon\eta}{2}} \left[ \frac{r(l - \varepsilon) + s(k + \varepsilon)}{2} \right]_{\varepsilon\eta} L_{r+s}^{(\eta k - \eta l + \varepsilon \eta)} + C_G \delta_{r+s, 0} , \quad (2.3)$$

where we define

$$[x]_+ = (q^x + q^{-x})/2 , \quad [x]_- = (q^x - q^{-x})/(q - q^{-1}) , \quad (2.4)$$

or in a symbolic way

$$[x]_\varepsilon = \left( \frac{2}{q - q^{-1}} \right)^{\frac{1-\varepsilon}{2}} \frac{q^x + \varepsilon q^{-x}}{2} . \quad (\varepsilon = \pm 1) \quad (2.5)$$

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<sup>1</sup>Their supercurrent generators are related to our  $G_r^{(k)}$  by the relation  $\mathcal{F}_r^{k, \pm} = \frac{1}{2}(G_r^{(-k)} \pm G_r^{(k)})$ .

The indices ( $n$  and  $k$ ) on  $L_n^{(k)}$  run over all integers. The lower index on  $G_r^{(k)}$  runs half-integers for the Neveu-Schwarz type algebra, and integers for the Ramond type. The upper index on  $G_r^{(k)}$  is an integer.

The central extensions  $C$  and  $C_G$  are given as follows [18]:

$$C = C_B + C_H , \quad (2.6)$$

with  $C_B$  for a scalar field,

$$C_B = \frac{1}{2} \sum_{j=1}^n [k(\frac{n}{2} - j)]_+ [l(\frac{n}{2} - j)]_+ [n - j]_- [j]_- , \quad (2.7)$$

and  $C_H$  for a Neveu-Schwarz (NS) fermion,

$$C_H = \frac{1}{2} \sum_{j=1}^n [k(\frac{n+1}{2} - j)]_- [l(\frac{n+1}{2} - j)]_- [n - j + \frac{1}{2}]_+ [j - \frac{1}{2}]_+ . \quad (2.8)$$

The  $C_H$  for a Ramond fermion is given by interchanging  $[x]_+$  and  $[x]_-$  in the expression  $C_B$ . The other one  $C_G$  is

$$C_G = \sum_{r \leq j < r+s} q^{(s/2+r-j)l + (r/2-j)k} [j - r]_- [j]_+ , \quad (2.9)$$

where  $j \in \mathbb{Z} + 1/2$  for the NS case, and  $j \in \mathbb{Z}$  for the R case.

In the following, we consider the scalar and fermion parts separately; i.e., splitting

$$L_n^{(k)} = H_n^{(k)} + B_n^{(k)} , \quad (2.10)$$

with

$$[H_n^{(k)}, B_m^{(l)}] = 0 . \quad (2.11)$$

These two parts,  $H_n^{(k)}$  and  $B_n^{(k)}$ , satisfy the algebra (2.1) with the central extensions  $C_B$  and  $C_H$  respectively, and the following set of decomposed relations is consistent with (2.2) and (2.3):

$$[H_n^{(k)}, G_r^{(l)}] = \frac{1}{2(q - q^{-1})} \sum_{\varepsilon, \eta = \pm 1} \varepsilon q^{\frac{n(l+\eta) - r(\varepsilon k + \eta)}{2}} G_{n+r}^{(\varepsilon k + l + \eta)} , \quad (2.12)$$

$$[B_n^{(k)}, G_r^{(l)}] = \frac{-1}{2(q - q^{-1})} \sum_{\varepsilon, \eta = \pm 1} \eta q^{\frac{-n(l+\eta) + r(\varepsilon k + \eta)}{2}} G_{n+r}^{(\varepsilon k + l + \eta)} , \quad (2.13)$$

$$\{G_r^{(k)}, G_s^{(l)}\} = \sum_{\varepsilon = \pm 1} \left( q^{\frac{r(l-\varepsilon) + s(k+\varepsilon)}{2}} B_{r+s}^{(k-l+\varepsilon)} + \varepsilon q^{\frac{-r(l-\varepsilon) - s(k+\varepsilon)}{2}} H_{r+s}^{(k-l+\varepsilon)} \right) + C_G \delta_{r+s,0} . \quad (2.14)$$

This is the explicit set of the BC superalgebra that we discuss in this paper. There are a few remarks: (i) for the consistency between Eqs.(2.12)–(2.14) and Eqs.(2.2) and (2.3), we need the properties

$$B_n^{(-k)} = B_n^{(k)} , \quad H_n^{(-k)} = -H_n^{(k)} . \quad (2.15)$$

(ii) The possible  $q \rightarrow 1$  limits of (2.1) are the following two ways:  $L_n^{(k)} \rightarrow L_n$  and  $L_n^{(k)} \rightarrow kL_n$ . Assuming the  $q \rightarrow 1$  limits to be

$$B_n^{(k)} \rightarrow L_n^B , \quad H_n^{(k)} \rightarrow kL_n^F , \quad (2.16)$$

where  $L_n^B$  and  $L_n^F$  are the usual Virasoro generators with  $c = 1$  and  $c = 1/2$ , the superalgebra (2.1) with Eqs. (2.12)–(2.14) reproduces the correct super Virasoro algebra. Thus, it is very natural to have *two different FFZ realizations* of the algebra (2.1), which satisfies the above limit property, as found in [8]. (iii) When realizing  $H_n^{(k)}$  in terms of fermionic field and related differential operators, the  $k = 0$  modes should be treated as in the limit of  $H_n^{(k)}/[k]_-$ . This seems to be natural from the above limit behavior of  $H_n^{(k)}$ . In contrast, as will be seen later, our FFZ realizations do not need this special treatment for the  $k = 0$  modes.

### 3 The free field realizations

In this section, using the field realizations, we derive various formulae for the BC superalgebra. Let us introduce the following deformed free fields ( $r \in \mathbb{Z} + \frac{1}{2}$  for NS and  $r \in \mathbb{Z}$  for R): the fermionic field

$$\Psi(z) = \sum_r \mathbf{b}_r z^{-r-\frac{1}{2}} , \quad \{\mathbf{b}_r, \mathbf{b}_s\} = [r]_+ \delta_{r+s,0} , \quad (3.1)$$

and the bosonic field

$$\Phi(z) = \sum_{n \in \mathbb{Z}} \mathbf{a}_n z^{-n-1} , \quad [\mathbf{a}_n, \mathbf{a}_m] = [n]_- \delta_{n+m,0} . \quad (3.2)$$

One can also express the bosonic field as

$$\Phi(z) = \frac{1}{z} [z \partial_z]_- \phi(z) , \quad (3.3)$$

if one introduces the analogue of a nondeformed massless scalar field,

$$\phi(z) = i\phi_0 + \mathbf{a}_0 \frac{q - q^{-1}}{2 \ln q} \ln z - \sum_{n \neq 0} \frac{\mathbf{a}_n}{[n]_-} z^{-n} , \quad [\phi_0, \mathbf{a}_n] = i\delta_{n,0} . \quad (3.4)$$

The generators satisfying all the relations presented in Section 2 are given by

$$H_n^{(k)} = \oint_0 \frac{dz}{2\pi i} z^{n+1} H^{(k)}(z) = \frac{1}{2} \sum_r \left[ k \left( \frac{n}{2} - r \right) \right]_- : \mathfrak{b}_r \mathfrak{b}_{n-r} : , \quad (3.5)$$

$$B_n^{(k)} = \oint_0 \frac{dz}{2\pi i} z^{n+1} B^{(k)}(z) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left[ k \left( \frac{n}{2} - j \right) \right]_+ : \mathfrak{a}_j \mathfrak{a}_{n-j} : , \quad (3.6)$$

$$G_r^{(k)} = \oint_0 \frac{dz}{2\pi i} z^{r+1/2} G^{(k)}(z) = \sum_{j \in \mathbb{Z}} q^{-k(\frac{r}{2}-j)} \mathfrak{a}_j \mathfrak{b}_{r-j} , \quad (3.7)$$

where the current fields are defined as

$$H^{(k)}(z) = \frac{1}{z(q - q^{-1})} : \Psi(q^{k/2}z) \Psi(q^{-k/2}z) : , \quad (3.8)$$

$$B^{(k)}(z) = \frac{1}{2} : \Phi(q^{k/2}z) \Phi(q^{-k/2}z) : , \quad (3.9)$$

$$G^{(k)}(z) = q^{-\frac{k}{4}} \Psi(q^{k/2}z) \Phi(q^{-k/2}z) . \quad (3.10)$$

We calculate the commutation relations between the generators and the fields:

$$[H_n^{(k)}, \Psi(z)] = z^n \left[ k \left( z\partial_z + \frac{n}{2} + \frac{1}{2} \right) \right]_- \left[ z\partial_z + n + \frac{1}{2} \right]_+ \Psi(z) , \quad (3.11)$$

$$[B_n^{(k)}, \Phi(z)] = z^n \left[ k \left( z\partial_z + \frac{n}{2} + 1 \right) \right]_+ \left[ z\partial_z + n + 1 \right]_- \Phi(z) , \quad (3.12)$$

$$\{G_r^{(k)}, \Psi(z)\} = z^{r+\frac{1}{2}} q^{-k(z\partial_z + \frac{r}{2} + 1)} \left[ z\partial_z + r + 1 \right]_+ \Phi(z) , \quad (3.13)$$

$$[G_r^{(k)}, \Phi(z)] = z^{r-\frac{1}{2}} q^{k(z\partial_z + \frac{r}{2} + \frac{1}{2})} \left[ z\partial_z + r + \frac{1}{2} \right]_- \Psi(z) . \quad (3.14)$$

We also find another set of commutator representations for an arbitrary parameter  $\Delta$ ;

$$[H_n^{(k)}, \frac{1}{z^{\Delta-1/2}} \Psi(z)] = z^n \left[ k \left( z\partial_z + \frac{n}{2} + \Delta \right) \right]_- \left[ z\partial_z + n + \Delta \right]_+ \frac{1}{z^{\Delta-1/2}} \Psi(z) , \quad (3.15)$$

$$[B_n^{(k)}, \frac{1}{z^{\Delta-1}} \Phi(z)] = z^n \left[ k \left( z\partial_z + \frac{n}{2} + \Delta \right) \right]_+ \left[ z\partial_z + n + \Delta \right]_- \frac{1}{z^{\Delta-1}} \Phi(z) , \quad (3.16)$$

$$\{G_r^{(k)}, \frac{1}{z^{\Delta-1/2}} \Psi(z)\} = z^r q^{-k(z\partial_z + \frac{r}{2} + \Delta)} \left[ z\partial_z + r + \Delta \right]_+ \frac{1}{z^{\Delta-1}} \Phi(z), \quad (3.17)$$

$$[G_r^{(k)}, \frac{1}{z^{\Delta-1}} \Phi(z)] = z^r q^{k(z\partial_z + \frac{r}{2} + \Delta)} \left[ z\partial_z + r + \Delta \right]_- \frac{1}{z^{\Delta-1/2}} \Psi(z), \quad (3.18)$$

where we notice that the differential operators acting on  $z^{\frac{1}{2}-\Delta}\Psi$  and  $z^{1-\Delta}\Phi$  on the r.h.s. coincide with matrix elements of a magnetic translation operator realization of the centerless BC superalgebra. (This will be shown in Section 4).

The following bilinear forms come from Eqs. (3.5)–(3.7):

$$H_n^{(k)} = -\frac{1}{2} \oint_0 \frac{dz}{2\pi i} : \Psi(z) \left( z^n \left[ k \left( z\partial_z + \frac{n}{2} + \frac{1}{2} \right) \right]_- \Psi(z) \right) :, \quad (3.19)$$

$$B_n^{(k)} = \frac{1}{2} \oint_0 \frac{dz}{2\pi i} : \Phi(z) \left( z^{n+1} \left[ k \left( z\partial_z + \frac{n}{2} + 1 \right) \right]_+ \Phi(z) \right) :, \quad (3.20)$$

$$\begin{aligned} G_r^{(k)} &= \oint_0 \frac{dz}{2\pi i} \Psi(z) \left( z^{r+\frac{1}{2}} q^{-k(z\partial_z + \frac{r}{2} + 1)} \Phi(z) \right) \\ &= \oint_0 \frac{dz}{2\pi i} \Phi(z) \left( z^{r+\frac{1}{2}} q^{k(z\partial_z + \frac{r}{2} + \frac{1}{2})} \Psi(z) \right). \end{aligned} \quad (3.21)$$

It is convenient to introduce a matrix form for these integral representations. If we define the matrix  $\mathcal{M}$  as

$$\mathcal{M} = \frac{1}{2} \oint_0 \frac{dz}{2\pi i} \chi Q \chi(z) \quad (3.22)$$

with

$$\chi(z) = \begin{pmatrix} \Phi(z) & 0 \\ 0 & \Psi(z) \end{pmatrix}, \quad Q(z) = \begin{pmatrix} z^{n+1} \left[ k \left( z\partial_z + \frac{n}{2} + 1 \right) \right]_+ & z^{r+\frac{1}{2}} q^{k(z\partial_z + \frac{r+1}{2})} \\ z^{r+\frac{1}{2}} q^{-k(z\partial_z + \frac{r}{2} + 1)} & -z^n \left[ k \left( z\partial_z + \frac{n+1}{2} \right) \right]_- \end{pmatrix}, \quad (3.23)$$

Eqs.(3.19)–(3.21) are simply described as

$$H_n^{(k)} = \mathcal{M}_{22}, \quad B_n^{(k)} = \mathcal{M}_{11}, \quad \frac{1}{2} G_r^{(k)} = \mathcal{M}_{12} = \mathcal{M}_{21}. \quad (3.24)$$

By using  $\chi$  and  $Q$ , the commutator representations (3.11)–(3.14) can be written in some useful integral forms. To show this, we notice that it is obvious from Eqs. (3.5)–(3.7) to have the following matrix relation

$$\mathcal{B}_\chi(w) \equiv \begin{pmatrix} [B_n^{(k)}, \Phi(w)] & [G_r^{(k)}, \Phi(w)] \\ \{G_r^{(k)}, \Psi(w)\} & [H_n^{(k)}, \Psi(w)] \end{pmatrix}^T = \oint \frac{dz}{2\pi i} \begin{pmatrix} z^{n+1} B^{(k)}(z) & z^{r+\frac{1}{2}} G^{(k)}(z) \\ z^{r+\frac{1}{2}} G^{(k)}(z) & z^{n+1} H^{(k)}(z) \end{pmatrix} \chi(w), \quad (3.25)$$



and the r.h.s. is evaluated as the OPE between  $\chi Q\chi(z)$  and  $\chi(w)$ . The OPE singular parts of the free fields are described as [18]

$$\Phi(z)\Phi(w) \sim \frac{1}{(z-qw)(z-q^{-1}w)} = \frac{1}{w} [w\partial_w]_- \frac{1}{z-w} , \quad (3.26)$$

$$\Psi(z)\Psi(w) \sim \frac{1}{2} \left( \frac{q^{\frac{1}{2}}}{z-qw} + \frac{q^{-\frac{1}{2}}}{z-q^{-1}w} \right) = [w\partial_w + \frac{1}{2}]_+ \frac{1}{z-w} , \quad (3.27)$$

or equivalently

$$\chi(z)\chi(w) \sim P(z) \frac{1}{z-w} ; \quad P(z) = \begin{pmatrix} -\frac{1}{z}[z\partial_z]_- & 0 \\ 0 & [z\partial_z + \frac{1}{2}]_+ \end{pmatrix} . \quad (3.28)$$

It is also convenient to use the following formulae (the analogues of partial integrations) for an analytic function  $f(z)$ ,

$$\oint \frac{dz}{2\pi i} z^n [k(z\partial_z + a)]_{\pm} f(z) = \pm \oint \frac{dz}{2\pi i} f(z) [k(z\partial_z - a + 1)]_{\pm} z^n , \quad (3.29)$$

$$\oint \frac{dz}{2\pi i} z^n q^{k(z\partial_z + a)} f(z) = \oint \frac{dz}{2\pi i} f(z) q^{-k(z\partial_z - a + 1)} z^n . \quad (3.30)$$

Then we obtain (3.25) in the following matrix form:

$$\mathcal{B}_{\chi}(w) = \oint_{wq^{\pm 1}} \frac{dz}{2\pi i} (\chi QP)(z) \frac{1}{z-w} . \quad (3.31)$$

Further applying (3.29) and (3.30) to each matrix element of (3.31), we also organize it into another different matrix form:

$$\begin{aligned} \mathcal{A}_{\chi}(w) &\equiv \begin{pmatrix} [B_n^{(k)}, \Phi(w)] & [G_r^{(k)}, \Phi(w)] \\ -\{G_r^{(k)}, \Psi(w)\} & [H_n^{(k)}, \Psi(w)] \end{pmatrix} = - \oint_w \frac{dz}{2\pi i} \frac{1}{z-w} (PQ\chi)(z) \\ &= -PQ\chi(w) . \end{aligned} \quad (3.32)$$

As shall be shown in Section 5, the differential operators  $QP$  and  $PQ$  appearing in the representations (3.31) and (3.32) are given by the elements of matrix realizations of the centerless BC superalgebra in terms of a generalized magnetic translation algebra. It is interesting to note that the matrix elements of  $P$  disappear in the bilinear forms (3.19)–(3.21), while in (3.31) and (3.32), the matrix  $P$  appears as the effect of field contractions.

## 4 The FFZ realizations

Magnetic translations satisfy the relation (with a suitable normalization)

$$T_{(k,n)} T_{(l,m)} = q^{\frac{ln-mk}{2}} (q - q^{-1})^{-1} T_{(k+l, n+m)} , \quad (4.1)$$

and this relation realizes the FFZ algebra [14]

$$[T_{(k,n)}, T_{(l,m)}] = \left[ \frac{ln - mk}{2} \right]_- T_{(k+l, n+m)} . \quad (4.2)$$

For the moment, we do not specify the forms (realizations) of  $T_{(k,n)}$ . In order to realize a superalgebra we also need the Grassmann operators:

$$\sigma^2 = (\sigma^\dagger)^2 = 0 , \quad \{\sigma, \sigma^\dagger\} = 1 , \quad (4.3)$$

which we regard as the quantities commuting with  $T_{(k,n)}$ ,

$$[T_{(k,n)}, \sigma] = [T_{(k,n)}, \sigma^\dagger] = 0 . \quad (4.4)$$

(In Section 5 we discuss the noncommuting case. Cf. Eq.(5.4).) Using the set of algebras (4.1)–(4.4), we find the following two realizations of the centerless BC superalgebra in accordance with  $\pm$  signs (we refer to them as  $\mathcal{R}^\pm$ ):

$$\hat{H}_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \eta q^{\pm \frac{\varepsilon n}{2}} T_{(\eta k + \varepsilon, n)} \sigma \sigma^\dagger , \quad (4.5)$$

$$\hat{B}_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \varepsilon q^{\pm \frac{\varepsilon n}{2}} T_{(\eta k + \varepsilon, n)} \sigma^\dagger \sigma , \quad (4.6)$$

$$\hat{G}_n^{(k)} = \sqrt{\frac{q - q^{-1}}{2}} \sum_{\varepsilon = \pm 1} q^{\pm \frac{\varepsilon n}{2}} \left( \varepsilon^{\frac{1 \pm 1}{2}} T_{(k + \varepsilon, n)} \sigma + \varepsilon^{\frac{1 \mp 1}{2}} T_{(-k + \varepsilon, n)} \sigma^\dagger \right) . \quad (4.7)$$

Now, we specify the  $T_{(k,n)}$  operators in terms of the differential operators, which possess an additional real parameter  $\Delta$ ; it is done by the correspondence

$$T_{(k,n)} \leftrightarrow z^n q^{-k(z\partial_z + \frac{n}{2} + \Delta)} / (q - q^{-1}) . \quad (4.8)$$

This is not literally the original magnetic translation operator on a two-dimensional surface, but actually the dimensionally reduced one [9]. It is convenient to rescale the Grassmann operators like

$$\sigma = \left( \sqrt{\frac{2}{q - q^{-1}}} \right)^{\pm 1} \hat{\sigma}_1 , \quad \sigma^\dagger = \left( \sqrt{\frac{q - q^{-1}}{2}} \right)^{\mp 1} \hat{\sigma}_2 , \quad \text{for } \mathcal{R}^\pm , \quad (4.9)$$

where  $\hat{\sigma}_1$  and  $\hat{\sigma}_2$  are still general Grassmann operators satisfying Eqs.(4.3) and (4.4). Later we will identify  $\hat{\sigma}_i$  with Pauli matrices or superspace operators as more concrete choices. We then have for

the realization  $\mathcal{R}^+$ ,

$$\hat{H}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_- \left[ z \partial_z + \Delta \right]_+ \otimes \hat{\sigma}_1 \hat{\sigma}_2 , \quad (4.10)$$

$$\hat{B}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_+ \left[ z \partial_z + \Delta \right]_- \otimes \hat{\sigma}_2 \hat{\sigma}_1 , \quad (4.11)$$

$$\hat{G}_n^{(k)} = -z^n q^{-k(z\partial_z + \frac{n}{2} + \Delta)} \left[ z \partial_z + \Delta \right]_- \otimes \hat{\sigma}_1 + z^n q^{k(z\partial_z + \frac{n}{2} + \Delta)} \left[ z \partial_z + \Delta \right]_+ \otimes \hat{\sigma}_2 , \quad (4.12)$$

and for the realization  $\mathcal{R}^-$ ,

$$\hat{H}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_- \left[ z \partial_z + n + \Delta \right]_+ \otimes \hat{\sigma}_1 \hat{\sigma}_2 , \quad (4.13)$$

$$\hat{B}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_+ \left[ z \partial_z + n + \Delta \right]_- \otimes \hat{\sigma}_2 \hat{\sigma}_1 , \quad (4.14)$$

$$\hat{G}_n^{(k)} = z^n q^{-k(z\partial_z + \frac{n}{2} + \Delta)} \left[ z \partial_z + n + \Delta \right]_+ \otimes \hat{\sigma}_1 - z^n q^{k(z\partial_z + \frac{n}{2} + \Delta)} \left[ z \partial_z + n + \Delta \right]_- \otimes \hat{\sigma}_2 . \quad (4.15)$$

Choosing  $\hat{\sigma}_i$  as the usual Pauli matrices  $\sigma_i$ ;  $i = 1, 2$ ,

$$\sigma_1 = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \sigma_x - i\sigma_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad (4.16)$$

and defining the differential operator matrix in each realization by

$$\mathcal{L}^\pm = \hat{H}_n^{(k)} + \hat{B}_n^{(k)} + \hat{G}_r^{(k)} , \quad \text{for } \mathcal{R}^\pm , \quad (4.17)$$

we can re-express them as

$$\mathcal{L}^+ = \mathcal{Q} \mathcal{P} , \quad \mathcal{L}^- = \mathcal{P} \mathcal{Q} , \quad (4.18)$$

with the matrices similar to  $Q$  and  $P$  (Cf. Eqs.(3.23) and (3.28))

$$\mathcal{P}(z) = \begin{pmatrix} -[z\partial_z + \Delta]_- & 0 \\ 0 & [z\partial_z + \Delta]_+ \end{pmatrix} , \quad (4.19)$$

$$\mathcal{Q}(z) = \begin{pmatrix} z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_+ & z^r q^{k(z\partial_z + \frac{r}{2} + \Delta)} \\ z^r q^{-k(z\partial_z + \frac{r}{2} + \Delta)} & -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_- \end{pmatrix} . \quad (4.20)$$

Introducing the following notations

$$\chi'(z) = \begin{pmatrix} \Phi'(z) & 0 \\ 0 & \Psi'(z) \end{pmatrix} , \quad \Phi'(z) = z^{1-\Delta} \Phi(z) , \quad \Psi'(z) = z^{\frac{1}{2}-\Delta} \Psi(z) , \quad (4.21)$$

we obtain the integral representations similar to (3.31) and (3.32):

$$\mathcal{B}_{\chi'}(w) = \oint \frac{dz}{2\pi i} (\chi' \mathcal{L}^+)(z) \frac{1}{z-w} \quad (\text{only for } \Delta = \frac{1}{2}) \quad (4.22)$$

$$\mathcal{A}_{\chi'}(w) = - \oint_w \frac{dz}{2\pi i} \frac{1}{z-w} (\mathcal{L}^- \chi')(z) = -(\mathcal{L}^- \chi')(w) . \quad (4.23)$$

Apparently, all the matrix elements of (4.23) represent the commutation relations (3.15)–(3.18).

We here put a remark on these two different realizations, which are understood as different orderings of  $\mathcal{P}$  and  $\mathcal{Q}$  in the matrix representations. The (differential operator) realization  $\mathcal{R}^+$  is regarded as a supersymmetric extension of the general differential operator expressions presented in [3]. However, the realization  $\mathcal{R}^-$  is not contained in the literature, and has a clear meaning as the adjoint representation (4.23). Note also that (4.22) holds only for  $\Delta = \frac{1}{2}$ , while (4.23) holds for arbitrary  $\Delta$ .

## 5 The generalized FFZ realizations

In order to obtain the differential operator realization corresponding to (3.32), which contains (3.11)–(3.14), it may rather be convenient to consider a different magnetic translation algebra. For this purpose, let us introduce an additional label on  $T_{(k,n)}$  correspondingly to the parameter  $\Delta$  as introduced in (4.8). We then consider the following generalized relations for the new sets of ‘magnetic translations’

$$T_{(k,n)}^\Delta T_{(l,m)}^{\Delta'} = \frac{q^{\frac{ln-mk}{2} + l(\Delta-\Delta')}}{q - q^{-1}} T_{(k+l, n+m)}^\Delta , \quad (5.1)$$

and this realizes the generalized FFZ algebra

$$[T_{(k,n)}^\Delta, T_{(l,m)}^{\Delta'}] = \frac{q^{\frac{ln-mk}{2} + l(\Delta-\Delta')}}{q - q^{-1}} T_{(k+l, n+m)}^\Delta - \frac{q^{-\frac{ln-mk}{2} + k(\Delta'-\Delta)}}{q - q^{-1}} T_{(k+l, n+m)}^{\Delta'} . \quad (5.2)$$

When  $\Delta = \Delta'$ , these reduce to the previous relations (4.1) and (4.2). We also need the Grassmann operators

$$\tilde{\sigma}^2 = (\tilde{\sigma}^\dagger)^2 = 0 , \quad \{\tilde{\sigma}, \tilde{\sigma}^\dagger\} = 1, \quad (5.3)$$

which however do not commute with  $T_{(k,n)}^\Delta$ :

$$T_{(k,n)}^\Delta \tilde{\sigma} = q^{-\frac{k}{2}} \tilde{\sigma} T_{(k,n)}^\Delta , \quad T_{(k,n)}^\Delta \tilde{\sigma}^\dagger = q^{\frac{k}{2}} \tilde{\sigma}^\dagger T_{(k,n)}^\Delta . \quad (5.4)$$

In this case, the above algebras enable us to have different realizations from  $\mathcal{R}^\pm$ , provided by a certain constraint on  $\Delta$  and  $\Delta'$ ; hence denoting them as  $\mathcal{R}_{(\Delta, \Delta')}^\pm$ . We find the following realizations:

First, we have the realization  $\mathcal{R}_{(\Delta, \Delta + \frac{1}{2})}^{\pm}$  with imposing  $\Delta' = \Delta + \frac{1}{2}$ ,

$$\tilde{H}_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \eta q^{\pm \frac{\varepsilon n}{2}} T_{(\eta k + \varepsilon, n)}^{\Delta} \tilde{\sigma} \tilde{\sigma}^{\dagger}, \quad (5.5)$$

$$\tilde{B}_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \varepsilon q^{\pm \frac{\varepsilon n}{2}} T_{(\eta k + \varepsilon, n)}^{\Delta'} \tilde{\sigma}^{\dagger} \tilde{\sigma}, \quad (5.6)$$

$$\tilde{G}_n^{(k)} = \sqrt{\frac{q - q^{-1}}{2}} \sum_{\varepsilon = \pm 1} q^{\pm \frac{\varepsilon n}{2}} \left( \varepsilon^{\frac{1 \pm 1}{2}} T_{(k + \varepsilon, n)}^{\Delta} \tilde{\sigma} + \varepsilon^{\frac{1 \mp 1}{2}} T_{(-k + \varepsilon, n)}^{\Delta'} \tilde{\sigma}^{\dagger} \right), \quad (5.7)$$

and secondly we have  $\mathcal{R}_{(\Delta, \Delta - \frac{1}{2})}^{\pm}$  with imposing  $\Delta' = \Delta - 1/2$ :

$$\tilde{H}_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \eta q^{\pm \frac{\varepsilon n}{2}} T_{(\eta k + \varepsilon, n)}^{\Delta} \tilde{\sigma}^{\dagger} \tilde{\sigma}, \quad (5.8)$$

$$\tilde{B}_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta = \pm 1} \varepsilon q^{\pm \frac{\varepsilon n}{2}} T_{(\eta k + \varepsilon, n)}^{\Delta'} \tilde{\sigma} \tilde{\sigma}^{\dagger}, \quad (5.9)$$

$$\tilde{G}_n^{(k)} = \sqrt{\frac{q - q^{-1}}{2}} \sum_{\varepsilon = \pm 1} q^{\pm \frac{\varepsilon n}{2}} \left( \varepsilon^{\frac{1 \pm 1}{2}} T_{(k + \varepsilon, n)}^{\Delta} \tilde{\sigma}^{\dagger} + \varepsilon^{\frac{1 \mp 1}{2}} T_{(-k + \varepsilon, n)}^{\Delta'} \tilde{\sigma} \right). \quad (5.10)$$

The differential operator realizations corresponding to these are obtained via the replacement

$$T_{(k, n)}^{\Delta} \leftrightarrow z^n q^{-k(z\partial_z + \frac{n}{2} + \Delta)} / (q - q^{-1}) \quad \text{for any } \Delta, \quad (5.11)$$

if we have the realizations of  $\tilde{\sigma}$  and  $\tilde{\sigma}^{\dagger}$ , which satisfy (5.3) and (5.4) respectively. Note that there are infinitely many realizations due to  $\Delta$ . We hence have to choose suitable realizations case by case, in order to make clear connections with other previously given realizations and with the  $q \rightarrow 1$  limits of known forms. In view of this, we only discuss the most interesting two cases, which are clearly related to the matrices  $PQ$  and  $QP$  presented in Section 2.

### 5.1 The $\mathcal{R}_{(\Delta, \Delta + \frac{1}{2})}^{-}$ case

In this case ( $\Delta' = \Delta + \frac{1}{2}$ ), we choose the operators satisfying (5.3) and (5.4) as

$$\tilde{\sigma} = z^{\frac{1}{2}} \sqrt{\frac{q - q^{-1}}{2}} \hat{\sigma}_1, \quad \tilde{\sigma}^{\dagger} = z^{-\frac{1}{2}} \sqrt{\frac{2}{q - q^{-1}}} \hat{\sigma}_2, \quad (5.12)$$

where  $\hat{\sigma}_i$  are the general operators, which satisfy the usual relations (4.3) and (4.4). Then the realizations (5.5)–(5.7) are reduced to

$$\tilde{H}_n^{(k)} = -z^n \left[ k \left( z\partial_z + \frac{n}{2} + \Delta \right) \right]_- \left[ z\partial_z + n + \Delta \right]_+ \otimes \hat{\sigma}_1 \hat{\sigma}_2, \quad (5.13)$$

$$\tilde{B}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta + \frac{1}{2} \right) \right]_+ \left[ z \partial_z + n + \Delta + \frac{1}{2} \right]_- \otimes \hat{\sigma}_2 \hat{\sigma}_1 , \quad (5.14)$$

$$\begin{aligned} \tilde{G}_n^{(k)} = & z^{n+\frac{1}{2}} q^{-k(z\partial_z + \frac{n}{2} + \Delta + \frac{1}{2})} \left[ z \partial_z + n + \Delta + \frac{1}{2} \right]_+ \otimes \hat{\sigma}_1 \\ & - z^{n-\frac{1}{2}} q^{k(z\partial_z + \frac{n}{2} + \Delta)} \left[ z \partial_z + n + \Delta \right]_- \otimes \hat{\sigma}_2 . \end{aligned} \quad (5.15)$$

If we employ (4.16) as a realization of  $\hat{\sigma}_i$ ;  $i = 1, 2$ , with choosing  $\Delta = \frac{1}{2}$  (and thus  $\Delta' = 1$ ), we find

$$L^-(z) = \tilde{H}_n^{(k)} + \tilde{B}_n^{(k)} + \tilde{G}_r^{(k)} = PQ(z) , \quad \text{for } \mathcal{R}_{(\frac{1}{2}, 1)}^- . \quad (5.16)$$

As announced below Eq.(3.32), each matrix element of  $L^-$  provides the adjoint representation in (3.32); i.e.,

$$\mathcal{A}_\chi(w) = -L^- \chi(w) . \quad (5.17)$$

We obtain the superspace realization of  $\mathcal{R}_{(\Delta, \Delta + \frac{1}{2})}^-$ , if we choose the ordinary Grassmann coordinate  $\theta$  and its derivative  $\partial_\theta$  as  $\hat{\sigma}_i$ :

$$\hat{\sigma}_1 = \partial_\theta , \quad \hat{\sigma}_2 = \theta . \quad (5.18)$$

The  $q \rightarrow 1$  limit should be taken in the combinations (q.v. Eq.(2.16)),

$$H_n^{(k)} / [k]_- + B_n^{(k)} \rightarrow L_n , \quad G_n^{(k)} \rightarrow G_n , \quad (5.19)$$

where each limit has a differential realization of the (centerless) super-Virasoro algebra:

$$L_n = -z^n \left( z \partial_z + \frac{n+1}{2} \theta \partial_\theta + \frac{n}{2} + \Delta \right) , \quad (5.20)$$

$$G_n = z^{n+\frac{1}{2}} (\partial_\theta - \theta \partial_z) - z^{n-\frac{1}{2}} \theta (n + \Delta) . \quad (5.21)$$

## 5.2 The $\mathcal{R}_{(\Delta, \Delta - \frac{1}{2})}^+$ case

In this case ( $\Delta' = \Delta - \frac{1}{2}$ ) we choose

$$\tilde{\sigma} = z^{\frac{1}{2}} \sqrt{\frac{q - q^{-1}}{2}} \hat{\sigma}_2 , \quad \tilde{\sigma}^\dagger = z^{-\frac{1}{2}} \sqrt{\frac{2}{q - q^{-1}}} \hat{\sigma}_1 , \quad (5.22)$$

and

$$\tilde{H}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta \right) \right]_- \left[ z \partial_z + \Delta \right]_+ \otimes \hat{\sigma}_1 \hat{\sigma}_2 , \quad (5.23)$$

$$\tilde{B}_n^{(k)} = -z^n \left[ k \left( z \partial_z + \frac{n}{2} + \Delta - \frac{1}{2} \right) \right]_+ \left[ z \partial_z + \Delta - \frac{1}{2} \right]_- \otimes \hat{\sigma}_2 \hat{\sigma}_1 , \quad (5.24)$$

$$\begin{aligned} \tilde{G}_n^{(k)} = & -z^{n-\frac{1}{2}} q^{-k(z\partial_z + \frac{n}{2} + \Delta - \frac{1}{2})} \left[ z \partial_z + \Delta - \frac{1}{2} \right]_- \otimes \hat{\sigma}_1 \\ & + z^{n+\frac{1}{2}} q^{k(z\partial_z + \frac{n}{2} + \Delta)} \left[ z \partial_z + \Delta \right]_+ \otimes \hat{\sigma}_2 . \end{aligned} \quad (5.25)$$

If we employ (4.16) as  $\hat{\sigma}_i$ , with  $\Delta = \frac{1}{2}$  (and thus  $\Delta' = 0$ ), we find for  $\mathcal{R}_{(\frac{1}{2},0)}^+$

$$L^+(z) = \tilde{H}_n^{(k)} + \tilde{B}_n^{(k)} + \tilde{G}_r^{(k)} = QP(z) . \quad (5.26)$$

As mentioned in the final paragraph of Section 3, each matrix element of  $L^+$  provides the adjoint representation in (3.31); i.e.,

$$\mathcal{B}_\chi(w) = \int \frac{dz}{2\pi i} (\chi L^+)(z) \frac{1}{z-w} . \quad (5.27)$$

The superspace realization of  $\mathcal{R}_{(\Delta,\Delta-\frac{1}{2})}^+$  can be obtained by identifying

$$\hat{\sigma}_1 = \theta , \quad \hat{\sigma}_2 = \partial_\theta , \quad (5.28)$$

and the  $q \rightarrow 1$  limits in (5.5)–(5.7) with (5.19) are given by

$$L_n = -z^n \left( z \partial_z + \frac{n+1}{2} \theta \partial_\theta + \Delta - \frac{1}{2} \right) , \quad (5.29)$$

$$G_n = z^{n+\frac{1}{2}} (\partial_\theta - \theta \partial_z) - z^{n-\frac{1}{2}} \theta (\Delta - \frac{1}{2}) . \quad (5.30)$$

In particular, setting  $\Delta = \frac{1}{2}$  (and  $\Delta' = 0$ ), we obtain the well-known form

$$L_n = -z^n \left( z \partial_z + \frac{n+1}{2} \theta \partial_\theta \right) , \quad (5.31)$$

$$G_n = z^{n+\frac{1}{2}} (\partial_\theta - \theta \partial_z) . \quad (5.32)$$

## 6 The ordinary field realizations

In this section we discuss two sets of the realizations of  $H_n^{(k)}$  and  $B_n^{(k)}$  in terms of ordinary free fields. In the first subsection, we present the set (referring to  $H_n'^{(k)}$  and  $B_n'^{(k)}$ ) which cannot be related to the deformed field realization by changing the normalizations of field oscillators. In this sense, this realization may sound to be nontrivial. The coefficients in the Sugawara forms only contain rational forms of  $[x]_\pm$ , and we hence call this type the *rational* realization in this paper.

On the other hand, the other set (presented in the second subsection) is related to the deformed fields by certain normalization changes. This type may hence be trivial, *however* these rescalings of field oscillators give rise to fractional differential operations on the ordinary free fields. Also, the coefficients in the Sugawara forms are irrational forms of  $[x]_\pm$ . We call this type the *irrational* realization.

## 6.1 The rational realizations

As can be seen in (5.17) and (5.27), the differential operators  $(L^\pm)_{ij}$ ;  $i = 1, 2$ , are the adjoint representations in the bases of the deformed fields (3.1) and (3.2). However, as seen in (3.22),  $(L^\pm)_{ij}$  themselves are not exactly the differential operators entered in the bilinear realizations (3.19)–(3.21) (Note that the matrix  $P$  drops out there). Contrastingly, we show that the  $P$  matrix part revives in the following particular free field bilinear realizations.

It is well known that by using the ordinary free fields,

$$\psi(z) = \sum_r d_r z^{-r-\frac{1}{2}}, \quad \{d_r, d_s\} = \delta_{r+s,0}, \quad (6.1)$$

$$\partial_z \varphi(z) = \sum_n a_n z^{-n-1}, \quad [a_n, a_m] = n \delta_{n+m,0}, \quad (6.2)$$

the Virasoro generators are given in the forms

$$L_n^F = -\frac{1}{2} \int dz \psi(z) (z^{n+1} \partial_z) \psi(z), \quad (6.3)$$

$$L_n^B = \frac{1}{2} \int dz \partial_z \phi(z) (z^{n+1} \partial_z) \phi(z). \quad (6.4)$$

We shall use the abbreviation  $\partial = \partial_z$  as long as it is clear. We find that the similar forms

$$H_n'^{(k)} = \frac{1}{2} \int dz \psi(z) (L^+)_{22} \psi(z), \quad (6.5)$$

$$B_n'^{(k)} = -\frac{1}{2} \int dz \partial \phi(z) (L^+)_{11} \phi(z) \quad (6.6)$$

satisfy the same algebra as (2.1) without changing the values of the central extensions given in the cases of  $H_n^{(k)}$  and  $B_n^{(k)}$  (see (2.7) and (2.8)). Obviously in these realizations the matrix elements of  $P$  participate in

$$H_n'^{(k)} = -\oint \frac{dz}{2\pi i} \frac{1}{2} \psi(z) z^n [z\partial + \frac{1}{2}]_+ [k(z\partial + \frac{1}{2}(n+1))]_- \psi(z) \quad (6.7)$$

$$= \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} [k(\frac{n}{2} - r)]_- [r]_+ : d_r d_{n-r} : , \quad (6.8)$$

and

$$B_n'^{(k)} = \frac{1}{2} \oint \frac{dz}{2\pi i} \partial \varphi(z) z^n [k(z\partial + \frac{n}{2})]_+ [z\partial]_- \varphi(z) \quad (6.9)$$

$$= \frac{1}{2} \sum_{l \in \mathbb{Z}} \frac{[l]_-}{l} [k(\frac{n}{2} - l)]_+ : a_l a_{n-l} : . \quad (6.10)$$



Applying the formula (3.29) to (6.7) and (6.9), we also derive another forms associated to  $(L^-)_{ii}$ :

$$H'_n{}^{(k)} = -\frac{1}{2} \oint \frac{dz}{2\pi i} L_{22}^- \psi(z) \cdot \psi(z) , \quad (6.11)$$

$$B'_n{}^{(k)} = -\frac{1}{2} \oint \frac{dz}{2\pi i} L_{11}^- \partial\phi(z) \cdot \phi(z) . \quad (6.12)$$

Here, the  $L^-$  can thus be understood as a partially integrated version of  $L^+$ . As we can see in (6.8) and (6.10),  $H'_n{}^{(k)}$  and  $B'_n{}^{(k)}$  are not related to (3.5) and (3.6) by changing any normalizations of the oscillators. Therefore, in this case, the deformed fields (3.1) and (3.2) cannot be interpreted in terms of a normalization change from the ordinary fields (6.1) and (6.2).

As a connection to the next section (irrational realizations), let us consider a fractional power decomposition of a  $q$ -derivative. For an analytic function  $f(x)$ , defining

$$f(z\partial)z^n \stackrel{\text{def.}}{=} f(n)z^n , \quad (6.13)$$

we observe

$$\begin{aligned} & \sqrt{[z\partial_z + \frac{1}{2}]_+} \sqrt{[w\partial_w + \frac{1}{2}]_+} < \psi(z)\psi(w) > = [z\partial_z + \frac{1}{2}]_+ < \psi(z)\psi(w) > \\ & = [w\partial_w + \frac{1}{2}]_+ < \psi(z)\psi(w) > , \end{aligned} \quad (6.14)$$

and

$$\begin{aligned} & \sqrt{\frac{[z\partial_z + 1]_-}{z\partial_z + 1}} \sqrt{\frac{[w\partial_w + 1]_-}{w\partial_w + 1}} < \partial\varphi(z)\partial\varphi(w) > = \frac{[z\partial_z + 1]_-}{z\partial_z + 1} < \partial\varphi(z)\partial\varphi(w) > \\ & = \frac{[w\partial_w + 1]_-}{w\partial_w + 1} < \partial\varphi(z)\partial\varphi(w) > = \frac{1}{w} [w\partial_w]_- < \partial\varphi(z)\varphi(w) > . \end{aligned} \quad (6.15)$$

Thus, roughly speaking, the pair of  $\psi(z)$  and  $[z\partial + \frac{1}{2}]_+ \psi(z)$  in (6.7), and the pair of  $\partial\phi(z)$  and  $z^{-1}[z\partial]_- \phi(z)$  in (6.9) may be replaced by bilinear forms of the following new fields:

$$\tilde{\Psi}(z) = \sqrt{[z\partial_z + \frac{1}{2}]_+} \psi(z) , \quad \tilde{\Phi}(z) = \sqrt{\frac{[z\partial_z + 1]_-}{z\partial_z + 1}} \partial\varphi(z) . \quad (6.16)$$

As shall be seen in the next subsection, this construction serves different realizations, however satisfying the same algebra (without changing the central extensions again). These fractional nonlocal operations (6.16) lead to simple normalization changes in the Fourier mode oscillators  $d_r$  and  $a_n$ .

## 6.2 The irrational realizations

Let us consider the deformed forms inferred through the argument from Eq.(6.14) to Eq.(6.16). First, it is very natural to define

$$H_n^{(k)} = -\frac{1}{2} \oint \frac{dz}{2\pi i} \tilde{\Psi}(z) z^n [k(z\partial + \frac{1}{2}(n+1))]_- \tilde{\Psi}(z) , \quad (6.17)$$

$$B_n^{(k)} = \frac{1}{2} \oint \frac{dz}{2\pi i} \tilde{\Phi}(z) z^{n+1} [k(z\partial + \frac{n}{2} + 1)]_+ \tilde{\Phi}(z) . \quad (6.18)$$

Easily understood from definitions (6.13) and (6.16), the Sugawara forms of these contain the irrational (root) forms of  $[x]_{\pm}$ ; i.e.

$$H_n^{(k)} = \frac{1}{2} \sum_r \left[ k\left(\frac{n}{2} - r\right) \right]_- \sqrt{[r]_+ [n-r]_+} : d_r d_{n-r} : , \quad (6.19)$$

$$B_n^{(k)} = \frac{1}{2} \sum_j \left[ k\left(\frac{n}{2} - j\right) \right]_+ \sqrt{\frac{[j]_- [n-j]_-}{j(n-j)}} : a_j a_{n-j} : . \quad (6.20)$$

Obviously, these cannot be transformed into the rational realizations (6.8) and (6.10) by performing  $d_r \rightarrow f(n, r) d_r$  etc. with any function  $f$ .

It is worth noticing that the bilinear field forms for these realizations are certainly the same as the deformed ones (3.8) and (3.9):

$$H^{(k)}(z) = \sum_n H_n^{(k)} z^{-n-2} = \frac{1}{z(q - q^{-1})} \tilde{\Psi}(q^{k/2} z) \tilde{\Psi}(q^{-k/2} z) , \quad (6.21)$$

$$B^{(k)}(z) = \sum_n B_n^{(k)} z^{-n-2} = \frac{1}{2} \tilde{\Phi}(q^{k/2} z) \tilde{\Phi}(q^{-k/2} z) . \quad (6.22)$$

In fact, Eqs.(6.19) and (6.20) coincide with the rescaled objects obtained from (3.5) and (3.6), if identified by the relations

$$\sqrt{[r]_+} d_r = \mathbf{b}_r , \quad \sqrt{\frac{[n]_-}{n}} a_n = \mathbf{a}_n , \quad (6.23)$$

where  $\mathbf{b}_r$  and  $\mathbf{a}_n$  are the deformed oscillators defined in (3.1) and (3.2). This identification also matches with the supercurrent generators  $G^{(k)}(z)$ , which are given by

$$G^{(k)}(z) = q^{-\frac{k}{4}} \tilde{\Psi}(q^{k/2} z) \tilde{\Phi}(q^{-k/2} z) = \sum_r G_r^{(k)} z^{-r-\frac{3}{2}} , \quad (6.24)$$

where

$$G_r^{(k)} = \sum_j q^{-k(\frac{r}{2}-j)} \sqrt{\frac{[j]_- [r-j]_+}{j}} a_j d_{r-j} . \quad (6.25)$$

The set of  $H_n^{(k)}$ ,  $B_n^{(k)}$  and  $G_r^{(k)}$  given in this realization of course satisfies the same superalgebra as given in Section 2. This is the reason why we have used the same notation as used in Section 3.

## 7 The $bc$ field representation

In this section, to find some more realizations for the algebra (2.1), we assume the general commutation relation (including the previous cases of deformed/undeformed bosons and fermions):

$$c_n b_m + \epsilon b_m c_n = D_n \delta_{n+m,0} , \quad (7.1)$$

where  $b_n$  and  $c_n$  are interpreted as the usual  $bc$  ghosts ( $h = 2$ ) for  $\epsilon = 1$ , and as the  $\beta\gamma$  superghosts ( $h = 3/2$ ) for  $\epsilon = -1$ . The previous bosons ( $h = 0$ ) and fermions ( $h = \frac{1}{2}$ ) are the  $c_n = b_n$  case. The  $h$  stands for the conformal dimension of the field corresponding to  $b_n$  (at  $q = 1$ ) in each case, and the normalization  $D_n$  should be given case by case.

Let us consider the following bilinear form with the coefficients  $s_n^l(k)$  yet to be determined from the consistency of (2.1):

$$E_n^{(k)} = \sum_l s_n^l(k) : b_{n-l} c_l : , \quad (l \in \mathbb{Z} - h). \quad (7.2)$$

We calculate the commutation relation for  $E_n^{(k)}$

$$\begin{aligned} [E_n^{(i)}, E_m^{(j)}] &= \alpha \sum_l \left( s_m^l(j) s_n^{l-m}(i) D_{l-m} - s_m^{l-n}(j) s_n^l(i) D_{l-n} \right) : b_{n+m-l} c_l : \\ &+ \alpha \epsilon \delta_{n+m,0} \sum_{0 < l \leq n} s_{-n}^{-l}(j) s_n^{n-l}(i) D_{n-l} D_{-l} , \end{aligned} \quad (7.3)$$

where  $\alpha = 2$  (or 1) for the previous boson/fermion cases (otherwise).

Examining this formula for the deformed  $bc$  and  $\beta\gamma$  cases with  $D_n = [n]_+$ , we find that the following form

$$E_n^{(k)} = - \sum_{l \in \mathbb{Z} - h} [k(\frac{n}{2} - l)]_- : b_{n-l} c_l : \quad (7.4)$$

satisfies the algebra (2.1) with the central extensions given by

$$C_E = \epsilon \sum_{0 < l \leq n} [j(\frac{n}{2} - l)]_- [i(\frac{n}{2} - l)]_- [l - n]_+ [l]_+ . \quad (7.5)$$

On the analogy of (3.2), instead of (7.4), it is also obvious to have different realizations with choosing  $D_n = [n]_-$ ;

$$E_n^{(k)} = \sum_{l \in \mathbb{Z} - h} [k(\frac{n}{2} - l)]_+ : b_{n-l} c_l : , \quad (7.6)$$

however the values of central extensions differ from the above case

$$C_{E'} = \epsilon \sum_{0 < l \leq n} [j(\frac{n}{2} - l)]_+ [i(\frac{n}{2} - l)]_+ [l - n]_- [l]_- . \quad (7.7)$$

One may further obtain some ordinary field realizations (without making any changes in the central extensions), examining the formula (7.3) with setting  $D_n = 1$ . For example, we find a third boson realization for  $B_n^{(k)}$ ,

$$B_n''^{(k)} = \frac{1}{2} \sum_{l \in \mathbb{Z}} [k(\frac{n}{2} - l)]_+ [l]_- : \alpha_l \alpha_{n-l} : \quad (7.8)$$

with

$$[\alpha_n, \alpha_m] = \delta_{n+m,0} , \quad (7.9)$$

which is related to Eq.(6.2) by the normalization relation

$$a_n = \sqrt{|n|} \alpha_n . \quad (7.10)$$

One can see that this realization,

$$B_n''^{(k)} = \frac{1}{2} \sum_{l \in \mathbb{Z}} [k(\frac{n}{2} - l)]_+ \frac{[l]_-}{\sqrt{|l(n-l)|}} : a_l a_{n-l} : , \quad (7.11)$$

differs from the previous realizations (6.10) and (6.20).

In the similar way to (7.8), we find the realizations

$$E_n^{(k)} = - \sum_{l \in \mathbb{Z}-h} [k(\frac{n}{2} - l)]_- [l]_+ : b_{n-l} c_l : , \quad \text{with } D_n = 1 \quad (7.12)$$

$$E_n'^{(k)} = \sum_{l \in \mathbb{Z}-h} [k(\frac{n}{2} - l)]_+ [l]_- : b_{n-l} c_l : , \quad \text{with } D_n = 1. \quad (7.13)$$

However, these are equivalent to (7.4) and (7.6) after rescaling not  $b_n$  but  $c_n \rightarrow [n]_{\pm} c_n$  respectively.

This is a different feature from the boson and fermion cases.

## 8 Conclusions and discussions

We have studied various realizations for the superalgebra proposed by Belov and Chaltikian [4] as a deformation of the Virasoro algebra in terms of: Moyal-like operators, differential operators, the adjoint commutators and their matrix representations, the bilinear integral forms of deformed/nondeformed fields. In Sections 4 and 5, we first have constructed the deformed (centerless) Virasoro operators at the abstract level based on the noncommuting Moyal type operators, and then have transformed them into differential operators realizations (in other words,  $q$ -differences

or reduced magnetic translations). The superspace realizations are straightforward. All these are an infinite number of realizations in accordance with a parameter  $\Delta$ . At these differential operator levels, we do not have to specify the value of  $\Delta$ .

While considering a connection of them to realizations in field theory, it has been necessary to specify  $\Delta$  (Sections 5 and 6). The adjoint matrix forms derived in Section 3 and the differential operator forms  $L^\pm$  in Section 5 have played a key role to find this connection, where the same  $PQ$  matrix combinations appear. In this way, all the formulae presented in Section 3 are reproduced for special values of  $\Delta$ . The similar statement holds for the results of Section 4, however, it is interesting to note that (4.23) holds without specifying the  $\Delta$  value.

Let us compare in details the results between Sections 4 and 5. In Section 4, we treated the Grassmann variables as quantities commuting with the FFZ parts, while in Section 5 we treated them as those noncommuting with the FFZ parts. This noncommutativity is realized by the Pauli matrices multiplied by  $z^{\pm 1/2}$  in the differential realizations (see (5.12)), while these multipliers are absorbed in the field definitions (4.21) in the former case. This is the reason why we encounter the unusual versions of adjoint commutator representations (3.15)–(3.18) (Cf. Eqs. (3.11)–(3.14)). Due to this fact, the  $PQ$  and  $\mathcal{P}Q$  matrix formulations become slightly different from each other (qv. (3.31), (3.32), (4.22) and (4.23)). These two sets of realizations are very similar at this stage, however it is contrast that their original abstract constructions are very different. As mentioned above, the arbitrariness of  $\Delta$  in (4.23) is a big difference as well.

Although we did not mention explicitly that the bilinear form (3.22) can be reproduced from our differential operators, it is also possible to do. Once we read the matrix  $Q(z)$  from  $L^\pm(z)$  by removing  $P(z)$ , we have only to sandwich  $Q$  by a couple of deformed/undeformed fields. This structure can be observed from (6.17) and (6.18) in the case of undeformed field case. The deformed field case is straightforward. The similar statement holds for the results of Section 4 with taking account of the  $z$  multiplications into the  $\chi'$  fields. After all, the matrix factor  $P$  plays a role of deforming propagators between two fields.

In Section 6, we have found the realizations, making use of the usual (nondeformed) free fields. The relations of the irrational realizations to the deformed field realizations are simply normalizations at the level of oscillators, however they are nontrivial at the level of fields. This fact suggests that a noncommutative string could be described in terms of either simple deformed fields, or highly nontrivial operations on ordinary fields. Judging from these observations, the relations among the

deformed field and their corresponding FFZ and differential realizations are the most significant of all the results we found in this paper. There then comes a question whether a physical system satisfying the relations (5.1)–(5.4) exists or not.

Concerning Section 7, we obtained the same copies of the BC superalgebra in terms of (deformed/nondeformed) ghost oscillators with new central extensions. In a  $n$  dimensional system like a string theory, the cancellation of the central extensions  $n(C_B + C_H) + C_E = 0$  is possible for a certain value of  $q$ . In the language of string theory,  $n$  means the critical dimensions, where anomalies vanish. Unfortunately the present algebra have a problem to apply this argument; i.e., the  $q \rightarrow 1$  limit is different from the usual Virasoro operators for the ghost parts. However, the idea of vanishing anomaly at a special value of  $q$  has become more promising as a model of deformed (noncommutative) superstring than ever. Similar applications of the central charges would be possible in any other models possessing the Virasoro algebra.

It is clear that our deformation is related to the Moyal quantization since we started from the FFZ realizations. In this sense also, the present deformed super Virasoro algebra should thus be understood as a noncommutative deformation of the original Virasoro algebra. We believe that it is important to further pursue the relations between our results and recent developing noncommutative physics and field theories. The representation theory should also be studied along the similar line.

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